CONCURSUL INTRAJUDEȚEAN DE MATEMATICĂ
„VASILE DUMITRACHE”
31 MAI 2008
Clasa a XI-a

I. Fie şirul $(x_n)_{n\geq 1}$ $x_n = \begin{vmatrix} a+x & a & \dots & a \\ a & a+x & \dots & a \\ \dots & \dots & \dots & \dots \\ a & a & \dots & a+x \end{vmatrix}$, a$\in$ R, determinant de ordin n.

Calculați:

$$\lim_{n\to\infty}\frac{1}{n}\left[\lim_{x\to 0}\left(\frac{1}{x}\sum_{i=1}^{n}\frac{x_{i+1}}{x_i}\right)\right].$$

Soluție:

x_n =(x+na)$\begin{vmatrix} 1 & 1 & \dots & 1 \\ a & a+x & \dots & a \\ \dots & \dots & \dots & \dots \\ a & a & \dots & a+x \end{vmatrix}$ =(x+na)$\begin{vmatrix} 1 & 0 & \dots & 0 \\ a & x & \dots & 0 \\ \dots & \dots & \dots & \dots \\ a & 0 & \dots & x \end{vmatrix}$ =(x+na)x^{n-1} (3 puncte)

Atunci $\dfrac{x_{i+1}}{x_i}=\dfrac{x^i\left[x+(i+1)a\right]}{x^{i-1}(x+ia)}=\dfrac{x\left[x+(i+1)a\right]}{x+ia}$. (1 punct)

Daca a=0 atunci limita este egala cu 1. (1 punct)

Daca a$\neq$ 0 $\lim\limits_{x\to 0}\dfrac{1}{x}\sum\limits_{i=1}^{n}\dfrac{x_{i+1}}{x_i}=\lim\limits_{x\to 0}\sum\limits_{i=1}^{n}\dfrac{x+(i+1)a}{x+ia}=\sum\limits_{i=1}^{n}\dfrac{i+1}{i}$ =n+ $\sum\limits_{i=1}^{n}\dfrac{1}{i}$ (2 puncte)

Din criteriul lui Stolz rezulta $\lim\limits_{n\to\infty}\dfrac{1}{n}\sum\limits_{i=1}^{n}\dfrac{1}{i}$ =0. (2 puncte)

Atunci limita ceruta este egala cu 1.(1 punct)

II. Determinați X$\in$ M_2 (R) dacă: $X^5=\begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$

Soluție:

Fie X = $\begin{pmatrix} a & b \\ c & d \end{pmatrix} a,b,c,d\in R$

$X^5 = \begin{pmatrix} ١ & ٣ \\ ٢ & ٦ \end{pmatrix}$ => $\det X = 0$ (1p.) => $X^2 = (a + d)X$ (2 p.) => $X^5 = (a + d)^4 X$ (2 p) => $\begin{pmatrix} ١ & ٣ \\ ٢ & ٦ \end{pmatrix} = (a + d)^4 \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ (1 p) => $7 = (a + d)^4 \, TrX$ (2 p) => $7 = (a + d)^5$ => $a + d = \sqrt[5]{7}$ (1 p) => $X = \frac{١}{\sqrt[٥]{٧^{٤}}} \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$ (1 p).

III. Fie funcția $f : R \to R$, $f(x) = ax^3 + px + q$ ($a \neq 0$).
Dacă f admite un maxim şi un minim notate respectiv cu M şi m demonstrați că:

$$M \cdot m = q^{٢} + \frac{٤p^{٣}}{٢٧a}$$

Solutie: Fie $M = f(x_1)$, $m = f(x_2)$, unde x_1, x_2, sunt radacinile ecuatiei $f'(x) = 0$, adica $3ax^2 + p = 0$. (1p)

Prin urmare:

$3ax_1^2 + p = 0$, $3ax_{٢}^{٢} + p = 0$, $ax_{١}^{٣} = -\frac{1}{3}px_1$ si $ax_2^3 = -\frac{١}{٣}px_{٢}$ (1 p)

Deci $M = f(x_1) = ax_{١}^{٣} + px_1 + q$, $m = f(x_2) = ax_{٢}^{٣} + px_2 + q$. (1 p)

Rezulta: $M = \frac{٢}{٣}px_{١} + q, m = \frac{٢}{٣}px_{٢} + q$, (1 p)

deci $M \cdot m = \left(\frac{٢}{٣}px_{١} + q\right)\left(\frac{٢}{٣}px_{٢} + q\right) = q^2 + \frac{2}{3}pq \cdot (x_1 + x_2) + \frac{4}{9}p^2x_1x_2$. (3p)

Dar: $x_1 + x_2 = 0$ si $x_1x_2 = \frac{p}{3a}$. (1p) Rezulta ca:

$M \cdot m = q^2 + \frac{2}{3}pq \cdot 0 + \frac{4}{9}p^2 \cdot \frac{p}{3a} = q^2 + \frac{4p^3}{27a}$. (2p)